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Equal Surplus Sharing in Grey Inventory Games

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Abstract

This study introduces a model where inventory costs are represented as grey numbers, rather than traditional crisp or stochastic values. Utilizing grey calculus, game-theoretic solutions are reinterpreted to address interval uncertainty within cooperative grey inventory games. Grey equal distribution rules are established for fair cost allocation. The model parameters are determined to construct a grey inventory game, which is applied to three shotgun companies in Türkiye. The calculated grey inventory costs and different game-theoretic solutions are presented. This study extends solutions like the Banzhaf value, *CIS*-value, *ENSC*-value, and *ED*-solution by incorporating interval uncertainty.

Keywords Grey numbers, Cooperative game theory, Inventory management, Equal surplus sharing rules

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1. Introduction

Inventory management studies aim to minimize the average total cost per unit time and determine the optimal order quantity for stocked materials. This problem was first addressed by Harris (1913) with the introduction of the Economic Order Quantity (EOQ) Model. Inventory management and the distribution of ordering and holding costs are critical in the business world. When firms place their orders simultaneously, total costs can be significantly reduced, but distributing these costs among firms is not straightforward.

Recent studies have focused on cooperative inventory games, where firms collaborate to minimize joint inventory costs. Recent studies focus on cooperative inventory games, where firms collaborate to minimize joint inventory costs. Meca (2004) extends inventory management to include cooperative scenarios, introducing proportional division mechanisms for cost sharing. They proposed a method to allocate costs fairly among participating firms. In another study, Meca (2006) introduced generalized holding cost games, which extend traditional models and study core-allocation families like the N-rational solution family, offering a new perspective on holding costs.

Centralized inventory management was discussed by Mosquera et al. (2007), who introduced the SOC-rule (Share the Ordering Cost) in inventory games. This rule is unique in its immunity to coalitional manipulation, ensuring fair cost distribution among cooperating firms. Further exploration of generalized holding cost games by Meca (2006) presented a new class of inventory games that further develop the theoretical framework for cost allocation.

The cost allocation procedure for the joint replenishment problem with first-order interaction was studied by Anily & Haviv (2007), adding to the understanding of cooperative strategies in inventory management. Dror & Hartman (2011) provided a comprehensive review of cooperative inventory games and their extensions within deterministic EOQ models, highlighting the evolution and impact of these cooperative strategies in inventory management.

Leng & Parlar (2009) analyze cost savings from sharing demand information in a three-level supply chain using cooperative game theory, ensuring stable and fair cost savings allocation. This study demonstrates the benefits of information sharing in supply chains. In a multi-agent inventory transportation system, Meca et al. (2010) examined cost sharing using an EOQ policy. They explored the conditions under which cooperation is profitable and how to ensure stable and fair cost allocations, focusing on the Shapley value and proportional cost sharing rules.

Karsten et al. (2017) analyzed cost allocation rules in cooperative games with elastic cost functions. They emphasized fairness criteria like coalitional rationality and benefit ordering, comparing various cost sharing rules such as proportional, serial cost sharing, and Shapley-esque rules. Olgun et al. (2017) introduced cooperative grey games to manage inventory costs where costs are represented as grey numbers. Applying grey system theory to shotgun companies in Turkey, the study demonstrates its effectiveness over traditional methods for handling imprecise information.

Kahraman & Aydemir (2020) developed a dual-objective inventory routing model using grey system theory to enhance logistics performance in medium-scale industrial distribution planning under uncertainty. Yang et al. (2021) proposed a coalition game model to promote fair cost distribution through the cooperative use of energy storage systems. Olgun & Aydemir (2021) introduced a cooperative game theory model where clients share warehouse space under capacity constraints to reduce costs and achieve savings. De & Mahata (2020) applied fuzzy game theory to develop an EOQ model for defective products, offering optimal solutions for inventory management under uncertainty. Guardiola et al. (2021) analyzed a new core allocation method aimed at providing stable cost-sharing solutions. Lastly, Liu et al. (2021) created a two-stage supply chain pricing model utilizing interval grey numbers to optimize collaborative pricing strategies, accounting for demand uncertainty and inventory costs.

Building on the results of Olgun et al. (2017), this paper focuses on a unified approach to a specific type of one-point solutions for cooperative games, known as equal surplus sharing solutions. These solutions include the Banzhaf value, the Centre-of-gravity of the Imputation-Set value (*CIS*-value), the Egalitarian Non-Separable Contribution value (*ENSC*-value), and the Equal Division solution (*ED*-solution) as discussed by Brink & Funaki (2008). The primary objective is to extend these solutions by incorporating interval uncertainty.

The *CIS*-value represents the individual worth of a player participating in the game as a solitary entity. The ENSC-value is derived from separable contributions, which are among the various marginal contributions a player can make. The *ENCIS*-value, or separable (marginal) contribution, indicates the value of a player's participation as a member of the player set. These values are well-established concepts in game theory literature, as noted by Driessen & Tijs (1985), Driessen (1988), Driessen & Funaki (1991), Driessen & Funaki (1994), Funaki (1998), Legros (1986), Moulin (1985).

In general, minimizing costs and accurately calculating stock levels in inventory management are crucial for enhancing operational efficiency within firms. However, in real-world scenarios, the uncertainty surrounding parameters related to inventory costs necessitates their estimation within predictable ranges. This study uses grey numbers to model these uncertainties, which forms the primary motivation for the research. Additionally, firms ordering similar products have the potential to collaborate to reduce inventory costs. In this context, a system has been developed that examines how costs can be distributed fairly when companies in the same industry place joint orders, enabling cost savings through collaboration.

This study offers an innovative contribution to the field by integrating inventory management and cooperative game theory with grey number theory. Specifically, it proposes three single-point solution concepts consistent with the Banzhaf value, *CIS*-value, *ENSC*-value, and ED-solution for the equitable distribution of the overall surplus. Three different cost allocation rules are modeled with grey numbers, providing a new perspective in the literature for fair and stable cost-sharing under uncertainty. This innovation makes a significant contribution for industries aiming to optimize cooperative strategies in environments where inventory costs are uncertain.

The real world is filled with various sources of uncertainty, including technological and market uncertainty, observation noise, experimental design limitations, incomplete information, and vagueness in decision-making processes. To address these challenges, cooperative grey games provide a useful game-theoretic framework for supporting decision-making in collaborative contexts that involve interval data.

Uncertain systems often emerge in natural settings, especially where small samples and incomplete information are present. Grey systems theory offers a powerful approach for handling these uncertainty problems, particularly with discrete data and incomplete information. Here, random variables are treated as grey numbers, while stochastic processes are considered grey processes. Thus, a grey system is one that contains information in the form of grey numbers, and a grey decision is a decision made within such a grey system.

The concept of grey interval uncertainty aligns naturally with real-world data, making grey interval numbers a popular structure in both theoretical models and software applications.

2. Preliminaries

2.1. Grey Numbers and Their Operators

In this study, tools for grey number calculations are essential, so we present several grey number concepts and operators used in this paper (Liu & Forrest, 2010).

A grey number is one whose exact value is not known, but the interval in which it resides is known. Specifically, a grey number is an element of a closed and bounded interval, meaning each grey number is within an interval $[\underline{a}, \overline{a}]$ where $\underline{a}, \overline{a} \in \mathbb{R}$. We denote the set of all closed and bounded intervals in $\mathbb{R}$ as $I(\mathbb{R})$. A grey number is represented as $\otimes \in [\underline{a}, \overline{a}]$,with $\underline{a}, \overline{a} \in \mathbb{R}$.

For instance, consider the temperature of a room which is between 20 and 25 degrees Celsius. The duration of a car trip is between 45 and 60 minutes. These two grey numbers can be written as: $\otimes_1 \in [20, 25]$, $\otimes_2 \in [45, 60]$, respectively.

Let $\otimes_1$ and $\otimes_2$ be two grey numbers where, $\otimes_1 \in [a, b], a < b$ and $\otimes_2 \in [c, d], c < d$.

Then,

- the sum of $\otimes_1$ and $\otimes_2$ is defined by

$$\otimes_1 + \otimes_2 \in [a + c, b + d] \tag{1}$$

- the multiplication of $\otimes_1$ with a positive scalar k is defined by

$$k \otimes_1 \in [ka, kb] \tag{2}$$

From point Eq. 1 and Eq. 2, it is evident that grey numbers exhibit a cone structure.

The whitenization value of a grey number $\otimes \in [a, b]$ is denoted by $\tilde{\otimes}$ and is defined as

$$\tilde{\otimes} = \alpha a + (1 - \alpha)b, \alpha \in [0, 1] \tag{3}$$

Typically, the whitenization value for a grey number is calculated by setting α to 1/2. This value is referred to as the equal weight mean whitenization value (Kose et al., 2011; Liu & Forrest, 2010).

2.2. Classical Cooperative Inventory Games

An inventory scenario is characterized by the triplet (N, a, m), where $N = \{1, ..., n\}$ presents the group of agents who have agreed to jointly place orders for a particular good, a denotes the fixed ordering cost, and $m = \{m_1, ..., m_n\}$ signifies the optimal number of orders for the firms. The inventory game that emerges from this inventory scenario (N, a, m) is denoted by $< N, c >$. Here, $c(S)$ represents the average inventory cost per unit time for the agents in subset S when they place their orders collectively, and is defined by $c(S) = 2a\sqrt{\sum_{i\in S} m_i^2}$ for each S. A classical inventory game is a cooperative game derived from an inventory scenario (Meca, 2004).

An n-person Economic Order Quantity (EOQ) model involves agents placing orders for goods with deterministic demand d_i and holding costs h_i. The average inventory cost per unit time, based on order size Q_i is given by $c(Q_i) = a\frac{d_i}{Q_i} + h_i\frac{Q_i}{2}$ the optimal order size $\hat{Q}_i$ is $\sqrt{\frac{2ad_i}{h_i}}$. When agents form a coalition S to minimize costs, the optimal order size for each agent is $\hat{Q}_i^* = \sqrt{\frac{2ad_i^2}{\sum_{j\in S} d_j h_j}}$, for all $i \in S$. The minimum average inventory cost for the coalition is $\frac{ad_i}{Q_i^*} + \sum_{j\in S}\frac{h_j Q_j^*}{2} = 2a\sqrt{\sum_{j\in S}\hat{m}_j^2}$. Cooperative inventory games can analyze EOQ model cooperation. The Share the Ordering Cost (SOC) rule, used for cost allocation is $(SOC)_i(N,c) = \frac{c^2(i)}{\sum_{j\in N} c^2(j)}c(N)$. This rule allocates costs based on the contribution of each agent to the total inventory cost.

3. Cooperative Grey Games

A grey cooperative game is defined as a pair $< N, c' >$, where $N = \{1, 2, ..., n\}$ is the set of players and $c' : 2^N \to \mathbb{R}$ is the characteristic function assigning a whitenized grey number to each coalition $S \in 2^N$. The value of each coalition S is given by $c'(S) = \tilde{\otimes}_S \in [\underline{a}_s, \overline{a}_s]$, with $\overline{a}_s, \underline{a}_s$representing the lower and upper bounds of the grey number in $I(\mathbb{R})$ such that $c'(\emptyset) = \tilde{\otimes}_0 \in [0, 0]$.

Example: Consider Example 4 from Alparskan Gök et al. (2011). In this scenario, players 1 and 2 each have a container they wish to store, while player 3 owns a storage facility with a capacity for one container. If player 1 stores his container, the benefit is $\otimes_1 \in [10, 30]$ and if player 2 stores his container, the benefit is $\otimes_2 \in [50, 70]$. The situation can be described as a cooperative grey game $< N, w' >$ with $N = \{1, 2, 3\}$, and $w'^{(S)} = \tilde{\otimes}_S \in [0, 0]$ if $3 \notin S, w'(\emptyset) = w'(3) = \tilde{\otimes}_3 = \tilde{\otimes}_0 = 0, w'(1, 3) = \tilde{\otimes}_{1,3} = 20$, and $w'(N) = w'(2, 3) = \tilde{\otimes}_{2,3} = 60$.

3.1. Cooperative Grey Inventory Games

The main criticism of the deterministic EOQ model is its assumption that parameter values are precisely known. Both cost parameters and demand rates are often uncertain.

Building on the results of Olgun et al. (2017), we use grey numbers for EOQ model parameters when they are not accurately known, except for the demand rate. The parameters are defined as:

- $\otimes_a \in [a_1, a_2]$: grey number for ordering cost
- $\otimes_h \in [h_1, h_2]$: grey number for holding cost
- $\hat{D}$: Deterministic demand rate
- $\widehat{AC}$: Average grey total inventory cost
- Q : Order quantity per period

The whitenized total cost function $\widehat{AC}(Q)$ for an order Q is $\widehat{AC}(Q) = (a_1 + a_2)\frac{\hat{D}}{2Q} + (h_1 + h_2)\frac{Q}{4}$. The optimum order size $\hat{Q}$ is $\hat{Q} = \sqrt{\frac{2(a_1+a_2)\hat{D}}{h_1+h_2}}$.

3.2. Grey ordering and holding situation

Olgun et al. (2017) define a grey inventory ordering situation as triple $< N, \otimes_a, \{m_i\}_{i\in N} >$ where $N = \{1, 2, .., n\}$ is the set of agents ordering together. $\otimes_a$ is the the grey ordering cost, and $m = \{m_1, ..., m_n\}$ represents the optimal number of orders. For a coalition S the optimal ordering cost is $\otimes_a \sqrt{\sum_{i\in S} m_i^2}$. A grey holding situation is characterized by the tuple $\left(N, \otimes_a, \left\{\otimes_{h_i}, d_i\right\}_{i\in N}\right)$, where

$N = \{1, ..., n\}$ represents a group of agents agreeing to jointly order a particular good. $\otimes_a$ is the fixed grey ordering cost, $\otimes_h$ is the grey holding cost and d_i is the demand rate of the i-th firm. Given a grey holding situation, the corresponding grey holding game $< N, c'_h >$ assigns to a coalition $S \subset N$ a minimal cost of $\sqrt{2 \otimes_a \otimes_{h_s} \sum_{i \in S} d_i}$ and $c'_h(\emptyset) = 0$.

If we substitute $Q_i = \frac{d_i Q_1}{d_1}$ for all $i \in S$. The cost can be expressed as a function of $Q_1 : \otimes_a \frac{d_1}{Q_1} + \sum_{i \in S} \otimes_{h_s} \frac{d_i Q_1}{2d_1}$. The cost is minimized when $Q_1 = \sqrt{\frac{2 \otimes_a d_1^2}{\otimes_{h_s} \sum_{j \in S} d_j}}$. Finally, the minimal cost per unit time for coalition S equals to $\sqrt{2 \otimes_a \otimes_{h_s} \sum_{i \in S} d_i}$.

4. On Equal Surplus Sharing

In this section, we introduce some game-theoretic solutions using grey calculus. Drawing inspiration from the works of Brink & Funaki (2008) and Olgun et al. (2017), we will reinterpret interval solutions with the grey calculus method specifically for the purpose of establishing grey equal distribution rules within the context of grey inventory games. Our goal, by leveraging these references, is to develop a comprehensive understanding of how grey calculus can be applied to create fair and balanced distribution rules in cooperative grey inventory scenarios.

The grey Banzhaf value, the grey *CIS*-value, grey *ENSC*-value, and grey *ED*-value solution, like the Grey Shapley value, is defined within the $(SMGG)^N$ class. This is because grey marginal operators are defined within the $(SMGG)^N$ class. Grey Shapley value is defined as $\phi' : (SMGG)^N \to G(\mathbb{R})^N$ and $\phi'_i(c') = \frac{1}{n!} \sum_{\sigma \in \prod(N)} \sum_{i \in N} m_i^\sigma(c') = \frac{1}{n!} n! c'(N) = c'(N) \in \left[\underline{a_N}, \overline{a_N}\right]$

Definition 4.1. *(The Grey Banzhaf value)* The Banzhaf value considers each player's entry into any coalition with equal probability. The Grey Banzhaf value is defined by $\beta' : \mathrm{SMGGN} \to G(\mathbb{R}^N)$ $\beta'_i(c') = \frac{1}{2^{|N|-1}} \sum_{i \in S} c'(S) - c'(S \setminus \{i\}), (c') \in \left[\underline{a_N}, \overline{a_N}\right]$ for all $i \in N$ and for all $c' \in \mathrm{SMGGN}$.

Definition 4.2. *(The Grey CIS-Value) GCIS*-value assigns each player their individual grey value and distributes the remainder of the grand coalition N, grey value equally among all players. The *GCIS*-value is defined by $\mathrm{GCIS}' : \mathrm{SMGGN} \to G(\mathbb{R}^N)$ and $|c'(N)| \leq \sum_{j \in N} c'(i) \mathrm{GCIS}'_i(c') = c'(\{i\}) + \frac{1}{|N|}\left(c'(N) - \sum_{j \in N} c'(\{j\})\right), (c') \in \left[\underline{a_N}, \overline{a_N}\right]$ for all $i \in N$ and for all $c' \in \mathrm{SMGGN}$.

Definition 4.3. *(The Grey ENSC- Value)* In a grey game from c' to $c'^* \in \mathrm{SMGG}^N$ class assigns to each coalition $S \subseteq N$ the grey value lost by the grand coalition N when coalition S separates from N. For each $S \subseteq N : c'^*(S) = c'(S) - c'(NS)$. The grey *ENSC*-value (*GENSC*-value) assigns the *CIS*-value of the dual game c'^* each game c'. $\mathrm{GENSC}' : \mathrm{SMGGN} \to G(\mathbb{R}^N)$ and $\mathrm{GENSC}'_i(c') = \mathrm{GCIS}'_i(c'^*) = \frac{1}{|N|}\left(c'(N) + \sum_{j \in N} c'(N \setminus \{j\})\right) - c'(N), (c') \in \left[\underline{a_N}, \overline{a_N}\right]$ for all $i \in N$ and for all $c' \in \mathrm{SMGGN}$. We find $|c'(N) + \sum_{j \in N} c'(N \setminus \{j\})| \leq |N\|c'(N \setminus \{i\})|$. Thus, the *GENSC*-value assigns each player's grey marginal contribution to the grand coalition and distributes the remainder equally among the players.

Definition 4.4. *(The Grey ED- Value)* The Grey *ED*-value (*GED*-value) assigns $\mathrm{GED}' : \mathrm{GGN} \to G(\mathbb{R}^N)$ and $\mathrm{GED}'_i(c') = \frac{c'(N)}{|} N|, (c') \in \left[\underline{a_N}, \overline{a_N}\right]$ for all $i \in N$ and for all $c' \in \mathrm{GGN}$.

5. An Application

In this section, we begin by determining the problem parameters to establish the foundational data for our analysis, subsequently utilizing these parameters to obtain a grey inventory game that models the cooperative interactions between the involved firms. Olgun (2017) define the weapon factories under consideration source their required barrels from the same supplier. Consequently, the annual ordering costs for the firms are identical and represented as the grey number $a'_1 \in [380, 400]$. The annual barrel demands for the firms are defined as follows: firm 1's demand $d'_1 \in [2500, 3000]$,firm 2's demand $d'_2 \in [1800, 2100]$ and firm 3's demand $d'_3 \in [1700, 1900]$. The annual holding costs are specified as follows: for firm 1 $h'_1 \in [0, 7, 0, 75]$, for firm 2 $h'_2 \in [0, 8, 0, 9]$ and for firm 3 $h'_3 \in [1, 1, 15]$. The grey holding cost parameters related to this study are summarized in Table 1.

Table 1. Grey inventory cost parameters

	Firm-1	Firm-2	Firm-3
Demand Rates (items/per year)	$d'_1 \in [2500, 3000]$	$d'_2 \in [1800, 2100]$	$d'_3 \in [1700, 1900]$
Ordering Costs (TL/year)	$a'_1 \in [380, 400]$	$a'_2 \in [380, 400]$	$a'_3 \in [380, 400]$
Holding Costs (TL/year)	$h'_1 \in [0, 7, 0, 75]$	$h'_2 \in [0, 8, 0, 9]$	$h'_3 \in [1, 1, 15]$

Source: Olgun et al. (2017)

The costs that the firms are responsible for individually, without cooperation, as well as the costs for when two firms and eventually all three firms collaborate, have been calculated and are presented in Table 2.

Table 2. Grey inventory costs of coalitions

$\{S\}$	$c'(\{S\})$
$\{1\}$	$c'(\{1\}) \in [1153, 3; 1341, 6]$
$\{2\}$	$c'(\{2\}) \in [1046, 1, 1229, 6]$
$\{3\}$	$c'(\{3\}) \in [1136, 7, 1322, 1]$
$\{12\}$	$c'(\{12\}) \in [1512, 5, 1749, 3]$
$\{13\}$	$c'(\{13\}) \in [1494, 8, 1714, 6]$
$\{23\}$	$c'(\{23\}) \in [1458, 8, 1697, 1]$
$\{123\}$	$c'(\{123\}) \in [1786, 6, 2049, 4]$

The Grey Banzhaf value of the game is illustrated as

$$\beta'_i(c') = \frac{1}{2^{|N|-1}} \sum_{i \in S} c'(S) - c'(S \setminus \{i\})(c') \in \left[\underline{a_N}, \overline{a_N}\right]$$

$$\begin{aligned}
\beta'_1(c') &= \tfrac{1}{2^2} \textstyle\sum_{1 \in S} c'(S) - c'(S \setminus \{1\}) \\
&= \tfrac{1}{2^2}(c'(1) + c'(12) - c'(2) + c'(13) - c'(3) + c'(123) - c'(23)) \\
&= [576.4, 651.525]
\end{aligned}$$

$$\begin{aligned}
\beta'_2(c') &= \tfrac{1}{2^2} \textstyle\sum_{2 \in S} c'(S) - c'(S \setminus \{2\}) \\
&= \tfrac{1}{2^2}(c'(2) + c'(12) - c'(1) + c'(23) - c'(3) + c'(123) - c'(13)) \\
&= [504.8, 586.775]
\end{aligned}$$

$$\begin{aligned}
\beta'_3(c') &= \tfrac{1}{2^2} \textstyle\sum_{3 \in S} c'(S) - c'(S \setminus \{3\}) \\
&= \tfrac{1}{2^2}(c'(3) + c'(13) - c'(1) + c'(23) - c'(2) + c'(123) - c'(12)) \\
&= [541.25, 615.675]
\end{aligned}$$

Then, The Grey Banzhaf value is

$\beta'(c') = ([576.4, 651.525], [504.8, 586.775], [541.25, 615.675])$.

The *GCIS*-value of the game is illustrated as

$$\mathrm{GCIS}'_i(c') = c'(\{i\}) + \tfrac{1}{|N|}\Big(c'(N) - \textstyle\sum_{j\in N} c'(\{j\}\Big))$$

$$\mathrm{GCIS}'_1(c') = c'(1) + \tfrac{1}{3}(c'(123) - (c'(1) + c'(2) + c'(3))$$
$$= [636.8, 726.967]$$
$$\mathrm{GCIS}'_2(c') = c'(2) + \tfrac{1}{3}(c'(123) - (c'(1) + c'(2) + c'(3))$$
$$= [529.6, 614.967]$$
$$\mathrm{GCIS}'_2(c') = c'(3) + \tfrac{1}{3}(c'(123) - (c'(1) + c'(2) + c'(3))$$
$$= [529.6, 614.967]$$

Then, the *GCIS*-value is

$$\mathrm{GCIS}'^{(c')} = ([636.8, 726.967], [529.6, 614.967], [529.6, 614.967]).$$

The *GENSC*-value of the game is illustrated as

$$\mathrm{GENSC}'(c') = -c'(N\setminus\{i\}) + \tfrac{1}{|N|}\Big(c'(N) + \textstyle\sum_{j\in N} c'(N\setminus\{j\})\Big)$$

$$\mathrm{GENSC}_1'^{(c')} = -c'(23) + \tfrac{1}{3}(c'(123) + c'(12) + c'(13) + c'(23))$$
$$= [625.433, 706.367]$$
$$\mathrm{GENSC}_2'^{(c')} = -c'(13) + \tfrac{1}{3}(c'(123) + c'(12) + c'(13) + c'(23))$$
$$= [589.433, 688.867]$$
$$\mathrm{GENSC}'_3(c') = -c'(12) + \tfrac{1}{3}(c'(123) + c'(12) + c'(13) + c'(23))$$
$$= [571.733, 654.167]$$

Then, the *GENSC*-value is

$$\mathrm{GENSC}'(c') = ([625.433, 706.367], [589.433, 688.867], [571.733, 654.167]).$$

The GED- value of the game is illustrated as

$$\mathrm{GED}'_i(c') = \tfrac{1}{|N|}c'(N)$$

$$\mathrm{GED}'_1(c') = \tfrac{1}{3}c'(\{1,2,3\}) = [595.533, 683.133]$$
$$\mathrm{GED}'_2(c') = \tfrac{1}{3}c'(\{1,2,3\}) = [595.533, 683.133]$$
$$\mathrm{GED}'_3(c') = \tfrac{1}{3}c'(\{1,2,3\}) = [595.533, 683.133]$$

Then, the GED- value is

$$\mathrm{GED}'(c') = ([595.533, 683.133], [595.533, 683.133], [595.533, 683.133]).$$

Table 3. Cooperative game model solution results for grey inventory costs of coalitions

	Firm 1	Firm 2	Firm 3
$\beta'(c')$	$[576.4, 651.525]$	$[504.8, 586.775]$	$[541.25, 615.675]$
$\mathrm{GCIS}'(c')$	$[636.8, 726.967]$	$[529.6, 614.967]$	$[529.6, 614.967]$
$\mathrm{GENSC}'(c')$	$[625.433, 706.367]$	$[589.433, 688.867]$	$[571.733, 654.167]$
$\mathrm{GED}'(c')$	$[595.533, 683.133]$	$[595.533, 683.133]$	$[595.533, 683.133]$

6. Conclusion

In this study, we introduce some game-theoretic solutions using grey calculus. Drawing inspiration from the works of Brink & Funaki (2008) and Olgun (2017). We reinterpret interval solutions with the grey calculus method specifically for the purpose of establishing grey equal distribution rules within the context of grey inventory games. By leveraging these references, along with the dual-objective inventory routing model based on grey system theory developed by Kahraman & Aydemir (2020) and the coalition game model proposed by Yang et al. (2021) for fair cost-sharing in energy management, we aim to develop a comprehensive understanding of how grey calculus can be applied to create fair and balanced distribution rules in cooperative grey inventory scenarios. The literature review shows that various approaches have been shown to reduce costs, enhance performance, and ensure fair cost-sharing in industrial and logistics planning processes under uncertainty.

Building on the findings of Olgun & Aydemir (2021), who successfully applied cooperative game theory to achieve cost savings among customers facing capacity constraints. Building on the results of Olgun et al. (2017), this paper focuses on a unified approach to a specific type of one-point solutions for cooperative games, known as equal surplus sharing solutions. These solutions include the Banzhaf value, the Centre-of-gravity of the Imputation-Set value (*CIS*-value), the Egalitarian Non-Separable Contribution value (*ENSC*-value), and the Equal Division solution (*ED*-solution) as discussed by Brink & Funaki (2008). The primary objective is to extend these solutions by incorporating interval uncertainty, as seen in methods like the EOQ model based on fuzzy game theory by De & Mahata (2020), which provides reliable solutions for inventory management of defective products under uncertainty. The *CIS*-value represents the individual worth of a player participating in the game as a solitary entity. The *ENSC*-value is derived from separable contributions, which are among the various marginal contributions a player can make. The *ENCIS*-value, or separable (marginal) contribution, indicates the value of a player's participation as a member of the player set. These values are well-established concepts in game theory literature, as noted by Driessen & Tijs (1985), Driessen & Funaki (1991); Driessen (1988), Driessen & Funaki (1994), Funaki (1998), Legros (1986), and Moulin (1985). The egalitarian division of the surplus from the overall profits results in three one-point solution concepts that correspond to the Banzhaf value, the *CIS*-value, *ENSC*-value, and *ED*-solution. This study aims to enhance these solutions by addressing interval uncertainty, thereby contributing to the broader understanding and application of these concepts in cooperative game theory.

We begin by determining the problem parameters to establish the foundational data for our analysis, subsequently utilizing these parameters to obtain a grey inventory game that models the cooperative interactions between the involved firms. The study by Guardiola et al. (2021), which proposes a new core allocation method for stable cost-sharing solutions, alongside the two-stage supply chain pricing model developed by Liu et al. (2021) for collaborative pricing under demand uncertainty, further enriches our approach in modeling these interactions under uncertainty. Olgun et al. (2017) define the weapon factories under consideration source their required barrels from the same supplier. Finally, an application is performed for three shotgun companies in Turkey. The calculated grey inventory costs for each firm and different game-theoretic solutions are presented in Table 3.

For future research, some possible model extensions may be considered, such as grey purchasing cost, grey allowance for stock outs, grey defective goods, and grey quantity discount situations. The applicability of these methods across various industrial fields confirms the effectiveness of grey system theory and game theory methods in uncertainty management. Future research could offer more advanced cost-sharing solutions by evaluating the performance of grey system and game theory models in broader uncertainty scenarios.

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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