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Research Article

Stochastic Contests with Linex Utility Functions

Pelin Canbolat, Ph.D.
Assist. Prof., Department of Industrial Engineering, Koc University, Istanbul, Turkey, pcanbolat@ku.edu.tr

* Department of Industrial Engineering, Koc University, Rumeli Feneri Yolu, 34450 Sariyer, Istanbul, Turkey

ABSTRACT

In this paper, we analyze the Nash equilibrium in a class of winner-takes-all stochastic contests among players with linear-exponential (linex) utility functions. In this contest, players are required to make upfront investments, which collectively determine their winning probabilities. We first show that a Nash equilibrium for such a contest exists and is unique, then set the equilibrium conditions, and study the properties of these conditions to gain insights into the structure of equilibrium. We show that the total equilibrium investment is bounded below and above, that the equilibrium has a cut characterization with respect to wealth, and that wealthier players invest more. The latter implies that richer is likely to get richer. For the special case with identical players, we show that an increase in the wealth or a decrease in the weight on the nonlinear component of the linex utility function results in an increase in the equilibrium investment.

Keywords: Game Theory, Stochastic Contests, Linear-Exponential Utility, Winner-Takes-All

Doğrusal-Üstel (Linex) Yarar Fonksiyonları ile Rassal Yarışmalar

ÖZ

Bu çalışmada tercihleri doğrusal-üstel yarar fonksiyonları ile belirlenen oyuncular arasında gerçekleşen ve tek kazananın olduğu yarışmaları incelenmektedir. Modelde oyuncuların yarışmanın başında yatırımlarını belirledikleri ve bütün bu yatırımların beraberce her bir oyuncunun kazanma olasılığını belirlediği varsayılmaktadır. Öncelikle bu modelin tek bir Nash dengesinin olduğunu gösterilmekte, ardından denge koşullarını belirlenip bu koşulların özellikleri incelenerek dengenin yapısı hakkında çıkarımlarda bulunulmaktadır. Bu çıkarımlar arasında dengede toplam yatırımın alt ve üst sınırlarının bulunduğu, yatırımların oyuncuların başlangıçtaki zenginliklerine göre bir kesi şeklinde gösterilebileceği, zengin oyuncuların daha fazla yatırım yaptıkları, dolayısıyla zenginlerin yarışmanın sonunda büyük olasılıkla daha da zengin olacakları yer almaktadır. Bunlar dışında, oyuncuların eşit zenginliğe sahip olduğu özel durumlarda zenginlikte bir artışın veya yarar fonksiyonunun doğrusal olmayan kısmının ağırlığında bir azalmanın yatırımlarda artışa neden olduğu kanıtlanmaktadır.

Anahtar Kelimeler: Doğrusal Üstel Yarar, Oyun Teorisi, Rassal Yarışmalar

1. Introduction

Canbolat et al. [3] analyzed the Nash equilibrium in a class of stochastic contests that are used to model competition of firms in research and development. They proposed an efficient method to compute the equilibrium, which turns out to be unique, and derived several managerial insights by exploring the functional properties of the equilibrium investments and utilities. Following this paper, Canbolat and Rothblum [4] studied the same model (with undiscounted rewards) under the assumption that players are risk averse and their preferences are represented via exponential utility functions. They showed the existence and uniqueness of the Nash equilibrium, and obtained a characterization that led to an efficient computational method. Both linear and exponential utility functions assume that preferences remain unchanged when wealth changes; that is, the individual will prefer the same alternative whether she is poor or rich. This aspect of linear and exponential utility functions, while making the analysis tractable, is subject to the criticism that in practice, the degree of risk sensitivity of most individuals varies as their wealth changes. More specifically, many individuals tend to be more tolerant to risk as they get wealthier. This paper aims to explore the implications of decreasing risk aversion in the context of stochastic contests with players that have \emph{linex} (linear-plus-exponential) utility functions.

Linex utility functions possess several properties that make them good candidates for representing preferences of individuals. Indeed, Bell [2] proved that if an individual's preferences over money can be represented by an increasing utility function, satisfy the axioms of expected utility, exhibit decreasing risk aversion, are close to risk neutral for small gambles for an extremely large wealth, and has one-switch property, then the utility function must be of the linex family. One-switch property means that the preference between any two gambles changes at most once as wealth increases. Later, Denuit et al. [8] showed that linex utility functions are also the only risk-averse utility functions that exhibit decreasing risk aversion and decreasing prudence in the stronger sense of Ross [10]. These properties make the linex utility function attractive from both theoretical and practical standpoints, and for this reason, this paper assumes that the players involved in the contest have preferences that can be represented via a linex utility function.

Stochastic contests considered in this paper form a special class of Tullock contests, introduced by Tullock [13] to model rent seeking. Cornes and Hartley explored the existence and the uniqueness of Nash equilibrium in Tullock contests, and obtained some comparative-statistics results under the assumption of constant absolute risk aversion in [6] and for more general utility functions in [7]. Yamazaki [14] showed the existence of a unique Nash equilibrium for general utility functions with decreasing absolute risk aversion and under budget constraints on investments. Differently from existing literature on Tullock contests with general risk-averse utility functions, the current paper focuses on linex utility functions, allows players to have different initial wealth, elicits the effects of the initial wealth and the weight on the nonlinear utility term on the Nash equilibrium.

The paper is organized as follows. Section 2 describes the model and introduces the notation. Section 3 defines and characterizes the Nash equilibrium. Section 4 explores further properties of this equilibrium in the case where players have the same wealth level, and Section 5 reports the observations of a numerical experiment for the contest between two players endowed with different wealth levels. Section 6 concludes the paper..

2. Model

Stochastic contests considered in this paper involve $n \geq 2$ players, who are identical except that their initial wealth may be different. At the beginning of the contest, each player makes an upfront investment to achieve a goal, e.g., development of a new product or technology. We let w_i be the initial wealth of player i and $x_i \geq 0$ be her investment. Players can borrow money if necessary, so it is possible to have $x_i > w_i$. The contest ends when a player reaches the goal. As in [3] and [4], we refer to a player with as an *active* player. We assume that the success probability of player is of the form

$$p_i(x) = \begin{cases} \dfrac{x_i}{\sum_{j=1}^{n} x_j} & if \sum_{j=1}^{n} x_j > 0, \\ 0 & if \sum_{j=1}^{n} x_j = 0. \end{cases} \tag{1}$$

where $x = (x_1, \ldots, x_n)$ is an *investment profile*. This probability can result for instance from the assumption that the time to reach the goal for player i follows exponential distribution with rate x_i and is independent of the completion times of other players, and the player that reaches the goal first wins the prize. This and more general contest success functions are discussed in [5] and [11].

We let the value of the contest prize be $R > 0$ for all players. The characterization of Nash equilibrium can be easily extended to asymmetric valuations of the prize by replacing R with R_i in the equilibrium condition of player i. Both [3] and [4] allow different R_i values, but in this paper, we assume players value the prize identically. We also assume that the utility function of each player for money is of the form

$$u(z) = (1-\alpha)z - \alpha e^{-\lambda z}, \tag{2}$$

where $0 < \alpha < 1$ and $\lambda > 0$. We note that $u(z)$ is the linear (risk-neutral) utility function for $\alpha = 0$ and the risk-averse exponential utility function with risk-sensitivity coefficient λ for $\alpha = 1$. For any $0 < \alpha < 1$,

$$u'(z) = 1 - \alpha + \alpha \lambda e^{-\lambda z} > 0,$$
$$u''(z) = -\alpha \lambda^2 e^{-\lambda z} < 0,$$
$$\rho(z) = -\frac{u''(z)}{u'(z)} = \frac{\alpha \lambda^2}{(1-\alpha)e^{\lambda z} + \alpha \lambda}.$$

The coefficient of absolute risk aversion $\rho(z)$ (as defined by Arrow [1] and Pratt [9]) is positive and decreasing in z, so u represents risk-averse preferences with decreasing absolute risk aversion. For increasing utility functions, decreasing absolute risk aversion implies prudence, which corresponds to the positivity of the third derivative of the utility function. For linex utility function (2),

$$u'''(z) = \alpha\lambda^3 e^{-\lambda z} > 0.$$

Additionally, the coefficient of absolute risk aversion $\rho(z)$ is increasing in α. As α increases, more weight is placed to the risk-averse part of (2), so preferences become more risk averse.

The expected utility of player i depends on her own investment as well as the investments of other players. Player i experiences the utility associated with terminal wealth $w_i - x_i + R$ if she wins the contests, and that of the terminal wealth $w_i - x_i$ if she loses. With winning probability (1) and linex utility (2), the expected utility of player i under the investment profile x is

$$\begin{aligned} U_i(x) &= p_i(x)u(w_i - x_i + R) + (1 - p_i(x))u(w_i - x_i) \\ &= (1-\alpha)\left[w_i - x_i + p_i(x)R \right] - \alpha e^{-\lambda(w_i - x_i)} \left[p_i(x)e^{-\lambda R} + 1 - p_i(x) \right]. \end{aligned}$$

In particular, for $x \neq 0$,

$$U_i(x) = (1-\alpha)\left[w_i - x_i + \frac{Rx_i}{\sum_{j=1}^{n} x_j} \right] - \alpha e^{-\lambda(w_i - x_i)} \left[\frac{e^{-\lambda R} x_i}{\sum_{j=1}^{n} x_j} + 1 - \frac{x_i}{\sum_{j=1}^{n} x_j} \right], \tag{3}$$

and $U_i(0) = u(w_i) = (1-\alpha)w_i - \alpha e^{-\lambda w_i}$

For notational simplicity, we let $S(x) = \sum_{i=1}^{n} x_i$ for any $x \in \mathbb{R}^n$, and $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. We also define $\beta = 1 - e^{-\lambda R}$ and note that $0 < \beta < 1$ for $R > 0$ and $\lambda > 0$. With this notation, (3) can be written as

$$U_i(x) = (1-\alpha)\left[w_i - x_i + \frac{Rx_i}{S(x)} \right] - \alpha e^{-\lambda(w_i - x_i)} \left[1 - \frac{\beta x_i}{S(x)} \right]. \tag{4}$$

To sum up, any instance of the stochastic contests considered in this paper can be described through the following data elements: number of players n, contest prize $R > 0$, player utility parameters $\lambda > 0$ and $\alpha \in (0,1)$, and player wealth levels $w_1, \ldots, w_n$. All these parameters are commonly known by all the players involved in the contest. The main purpose of the following sections is to shed light into the effect of α and $w_1, \ldots, w_n$ on the Nash equilibrium of these stochastic contests.

3. Nash Equilibrium

In the Nash equilibrium, each player makes an investment with the objective of maximizing her own expected utility. The investment $\bar{x}_i$ of player i is a *best response* of player i to x_{-i} if $U_i (\bar{x}_i, x_{-i}) \geq U_i (x_i, x_{-i})$ for all $x_i \geq 0$. An investment profile x^* is a *Nash equilibrium* if x_i^* is a best response to x_{-i}^* for every $i = 1, \ldots, n$. As shown in [3] and [4] for the special cases of linear and risk-averse exponential utilities, player i has no best response to $x_{-i} = 0$ under the linex utility function. To see why, suppose other players are not investing at all in the contest, then player i can guarantee winning by investing a very small amount, but since investment is a sunk cost, she will enjoy a greater utility by minimizing her investment without making it zero. By reducing her investment, her utility approaches $u(w_i + R) = (1-\alpha)(w_i + R) - \alpha e^{-\lambda(w_i + R)}$, but never attains this maximum, since when she invests zero, her probability of winning drops down to zero.

Cornes and Hartley [7] show the existence of a Nash equilibrium in general contests with contest success functions

$$p_i(x) = \frac{f_i(x_i}{\sum_{j=1}^{n} f_j(x_j)},$$

where f_i is continuous and twice continuously differentiable in $x_i \geq 0$, $f_i(0) = 0$, $f_i'(x_i) > 0$ for $x_i \geq 0$, and $f''(x_i) \leq 0$ for $x_i > 0$, and general increasing concave continuously differentiable utility functions $u_i(z)$. In our model, $f_i(x_i) = x_i$ and $u_i(z) = (1-\alpha)z - \alpha e^{-\lambda z}$, and these satisfy the conditions assuring the existence of a Nash equilibrium, so we conclude that the stochastic contests described in Section 2 have at least one Nash equilibrium. The following results are instrumental in the characterization of Nash equilibrium under linex utility function.

Lemma 1. Given $x_{-i} \neq 0$, the expected utility $U_i(x)$ of player i is differentiable and strictly concave in $x_i \geq 0$, and it converges to $-\infty$ as x_i converges to infinity.

Proof. Given $x_{-i} \neq 0$, differentiating (2) with respect to x_i gives

$$\frac{\partial U_i(x)}{\partial x_i} = (1-\alpha)\left(\frac{RS(x_{-i})}{S(x)^2} - 1\right) - \alpha e^{-\lambda(w_i - x_i)}\left(\lambda - \frac{\lambda \beta x_i}{S(x)} - \frac{\beta S(x_{-i})}{S(x)^2}\right). \quad (5)$$

The second derivative with respect to x_i is

$$\frac{\partial^2 U_i(x)}{\partial x_i^2} = -\frac{2(1-\alpha)RS(x_{-i})}{S(x)^3}$$
$$-\alpha e^{-\lambda(w_i - x_i)}\left\{\lambda^2(1-\beta) + \frac{\beta S(x_{-i})}{S(x)}\left[\left(\lambda - \frac{1}{S(x)}\right)^2 + \frac{1}{S(x)^2}\right]\right\} < 0,$$

where the inequality follows from the assumptions $\lambda > 0$, $R > 0$ (so $0 < \beta < 1$), and $0 < \alpha < 1$. Also, as x_i converges to infinity, $p_i(x) = x_i / S(x)$ converges to 1, so the bracketed expressions in (4) converge to $-\infty$ and $1-\beta$ respectively. The exponential term before the second bracketed expression converges to ∞; therefore, (4) converges to $-\infty$.

To simplify the notation and the proofs in the sequel, we define three functions and note their relevant properties as observations below.

Observation 1. The function $f(x) = (1-e-x)/x$ is decreasing in x>0 and converges down to 1 as x decreases to zero. The monotonicity follows from its first derivative

$$f'(x) = e^{-x}(x - e^x + 1)/x^2 < 0 \quad \text{forall } x > 0,$$

Since $e^x > 1 + x$, and the limit as x converges down to zero is

$$\lim_{x \downarrow 0} f(x) = \lim_{x \downarrow 0} \frac{e^{-x}(e^x - 1)}{x} = \lim_{x \downarrow 0} \frac{(x + x^2/2 + x^3/6 + \cdots)}{x} = 1.$$

These two properties imply that

$$f(x) > 1 \quad \text{forall } x > 0. \tag{6}$$

Observation 2. The function $g(x) = \lambda F - \lambda\beta Fx - \beta(1-x)$ is increasing in x if $\lambda F < 1$ and otherwise, for all $0 \le x \le 1$,

$$\begin{aligned} g(x) &= \lambda F - \lambda\beta Fx - \beta(1-x) > \lambda F - \lambda Fx - (1-x) \\ &= \lambda F - \left[\lambda Fx + (1-x)\right] \ge 0 \end{aligned} \tag{7}$$

where the first inequality holds because $\beta \in (0,1)$, and the last one holds because the bracketed expression is a convex combination of λF and 1, and $\lambda F \ge 1$, so the bracketed expression is less than or equal to λF.

Observation 3. The function $h_m(x) = (m-1)x + xe^{-x} - m + me^{-x}$ with $m \ge 2$ is increasing in $x \ge 0$, since

$$\begin{aligned} \frac{dh_m(x)}{dx} &= (m-1) + e^{-x} - xe^{-x} - me^{-x} \\ &= e^{-x}\left[(m-1)e^{x} + 1 - x - m\right] > e^{-x}\left[(m-1)(1+x) + 1 - x - m\right] \\ &= e^{-x}(m-2)x \ge 0. \end{aligned}$$

$$h_m(x) > h_m(0) = 0 \quad \text{forall } x > 0. \tag{8}$$

Lemma 2. Suppose $x_{-i} \ne 0$.

i. If $S(x_{-i}) \ge R$, then the best response of player i to x_{-i} is zero.

ii. If $\beta < \lambda S(x_{-i}) < \lambda R$ and

$$w_i \le -\frac{1}{\lambda}\ln\left\{\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{R - S(x_{-i})}{\lambda S(x_{-i}) - \beta}\right)\right\},$$

then the best response of player i to x_{-i} is zero.

iii. Otherwise, the best response of player i to x_{-i} is positive and it is the unique solution $x_i = \overline{x}_i$ of the nonlinear equation

$$(1-\alpha)\left(RS(x_{-i}) - S(x)^2\right) = \alpha e^{-\lambda(w_i - x_i)}\left(\lambda S(x)^2 - \lambda\beta x_i S(x) - \beta S(x_{-i})\right). \tag{9}$$

Proof. By Lemma 1., given $x_{-i} \ne 0$, $U_i(x)$ is strictly concave in $x_i \ge 0$ and converges to $-\infty$ as x_i converges to ∞, so it must have a finite maximizer $\overline{x}_i \ge 0$. Specifically, if the derivative of $U_i(x)$ with respect to x_i evaluated at $x_i = 0$ is nonpositive, the unique maximizer is zero; otherwise, it is positive and is the unique stationary point of $U_i(x)$. For $x_i = 0$, $S(x_{-i}) = S(x)$ and the derivative (5) is

$$\left.\frac{\partial U_i(x)}{\partial x_i}\right|_{x_i=0} = (1-\alpha)\left(\frac{R}{S(x_{-i})} - 1\right) - \alpha e^{-\lambda w_i}\left(\lambda - \frac{\beta}{S(x_{-i})}\right). \tag{10}$$

Based on Observation 1, $\beta / \lambda = Rf(\lambda R)$ and by (6), $\beta < \lambda R$. Accordingly, if $S(x_{-i}) \ge R$, then $\lambda S(x_{-i}) \ge \lambda R > \beta$, (10) is nonpositive. This is true also when $\beta < \lambda S(x_{-i}) < \lambda R$ and

$$w_i \le -\frac{1}{\lambda}\ln\left\{\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{R - S(x_{-i})}{\lambda S(x_{-i}) - \beta}\right)\right\}.$$

In both cases, the unique maximizer is zero.

On the other hand, if $\lambda S(x_{-i}) \le \beta$, then $S(x_{-i}) < R$, so (10) is positive. Also if $\lambda S(x_{-i}) > \beta$ and

$$w_i > -\frac{1}{\lambda}\ln\left\{\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{R-S(x_{-i})}{\lambda S(x_{-i})-\beta}\right)\right\},$$

(10) is positive. In both cases, the unique maximizer of $U_i(x)$ is the value $x_i = \bar{x}_i > 0$ that makes (5) zero. Equating (5) to zero and rearranging yields (9).

Proposition 1. The stochastic contest among players with linex utility functions has a unique Nash equilibrium.

Proof. The existence of a Nash equilibrium follows from Theorem 3.1 in [7] with $f_i(x_i) = x_i$ and $u_i(z) = (1-\alpha)z - \alpha e^{-\lambda z}$. We obtain the uniqueness from Theorem 4.2 in [7] by writing the derivative of $U_i(x)$ in terms of the total investment $F = S(x) > 0$ and $z = x_i / F$ as

$$\phi_i(F,z) = F\left\{(1-\alpha)\left[R(1-z)-F\right] - \alpha e^{-\lambda(w_i - Fz)}\left[\lambda F - \lambda\beta F z - \beta(1-z)\right]\right\},$$

and showing that for each i and $F > 0$, there exists a unique $0 \le z \le 1$ such that $\phi_i(F,z) \le 0$ and $z\phi_i(F,z) = 0$.

If $F \ge R$, then the first bracketed expression is nonpositive, and the second one is nonnegative, since

$$\lambda F - \lambda\beta F z - \beta(1-z) \ge (\lambda F - \beta)(1-z) \ge (\lambda R - \beta)(1-z) \ge 0,$$

where the last equality follows from $\beta / \lambda = Rf(\lambda R)$ and (6). Accordingly, $\phi_i(F,z) \le 0$ for all $0 \le z \le 1$. Also if $0 < z \le 1$, the first inequality is strict, so $\phi_i(F,z) < 0$. Hence the unique $0 \le z \le 1$ such that $\phi_i(F,z) \le 0$ and $z\phi_i(F,z) = 0$ is $z = 0$ in this case.

Suppose now $F < R$ and $\lambda F \le 1$. Then

$$\frac{\partial \phi_i(F,z)}{\partial z} = -F\left\{(1-\alpha)R + \alpha e^{-\lambda(w_i - Fz)}\left[(\lambda F - \beta)^2 + \beta(1-\beta) + \lambda\beta F z(1-\lambda F) < 0\right]\right\},$$

And $\phi_i(F,1) = -(1-\alpha)F - \alpha e^{-\lambda(w_i - F)}\lambda F(1-\beta) < 0$, so either there exists a unique $0 < z < 1$ such that $\phi_i(F,z) = 0$, in which case $\phi_i(F,0) > 0$, or $\phi_i(F,z) \le 0$ and $z = 0$ is the only $0 \le z \le 1$ such that $\phi_i(F,z) \le 0$ and $z\phi_i(F,z) = 0$.

Finally, consider the case where $F < R$ and $\lambda F > 1$. In this case, $\lambda F - \lambda\beta F z - \beta(1-z)$ is decreasing in z, so is minimized at $z = 1$, implying $\lambda F - \lambda\beta F z - \beta(1-z) \ge \lambda F(1-\beta) > 0$. The equation $\phi_i(F,z) = 0$ is then equivalent to

$$\frac{R(1-z)-F}{\lambda F - \lambda\beta F z - \beta(1-z)} = \frac{\alpha}{1-\alpha} e^{-\lambda(w_i - Fz)},$$

Differentiating the left-hand side with respect to z gives

$$\frac{-F(\lambda\beta F - \beta - \lambda\beta R + \lambda R)}{\left[\lambda F - \lambda\beta F z - \beta(1-z)\right]^2} < 0,$$

so the left-hand side is decreasing in z whereas the right-hand side is increasing in z . Accordingly, there can be at most one solution $0 \le z \le 1$ that satisfies this equality.

If there is no such solution, then $z=0$ is the only $0 \le z \le 1$ such that $\phi_i(F,z) \le 0$ and $z\phi_i(F,z)=0$; otherwise,

$$\begin{aligned}\phi_i(F,0) &= F\left\{(1-\alpha)[R-F]-\alpha e^{-\lambda(w_i)}[\lambda F-\beta]\right\}\\ &= F(1-\alpha)[\lambda F-\beta]\left\{\frac{R-F}{\lambda F-\beta}-\frac{\alpha}{1-\alpha}e^{-\lambda(w_i)}\right\}\\ &> F(1-\alpha)[\lambda F-\beta]\left\{\frac{R(1-z)-F}{\lambda F-\lambda\beta Fz-\beta(1-z)}=\frac{\alpha}{1-\alpha}e^{-\lambda(w_i-Fz)}\right\}=0,\end{aligned}$$

which completes the proof.

Theorem 1. In the unique Nash equilibrium x^* of the stochastic contest with parameters n,R,λ,α, and $w_1,\ldots,w_n$, at least two players are active, the total equilibrium investment $F=S(x^*)$ is in the interval $(0,R)$, and the investment of each active player i is the unique solution $x_i=x_i^*$ of the equation

$$(1-\alpha)\left[R(F-x_i)-F^2\right]=\alpha e^{-\lambda(w_i-x_i)}\left[\lambda F^2-\lambda\beta x_i F-\beta(F-x_i)\right]. \tag{11}$$

Furthermore,

i. If $\lambda F \le \beta$, then all players are active.

ii. If $\lambda F > \beta$, then only players with wealth

$$w_i > -\frac{1}{\lambda}\ln\left\{\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{R-F}{\lambda F-\beta}\right)\right\}$$

are active.

Proof. By Proposition 1., there exists a unique Nash equilibrium, and as argued at the beginning of this section, player i does not have a best response to $x_{-i}=0$, so an investment profile with no or only one active player cannot be a Nash equilibrium. Hence at least two players must be active in the unique Nash equilibrium x^*, so $F=S(x^*)>0$. We obtain (11) by replacing $S(x^*)$ with F and $S(x_{-i}^*)$ with $F-x_i^*$ in (9). Now suppose $F \ge R$, then the left-hand side of (11) is nonpositive since $R(F-x_i^*)-F^2 \le F(R-F) \le 0$. However, if also $\lambda F \ge 1$, then

$$\begin{aligned}&\lambda F^2-\lambda\beta x_i^* F-\beta(F-x_i^*)\\ &=\lambda F^2-\beta(\lambda F-1)x_i^*-\beta F \ge \lambda F^2-\beta(\lambda F-1)F-\beta F\\ &=\lambda F^2(1-\beta)>0;\end{aligned}$$

else, $\lambda R \le \lambda F < 1$ and $\beta < \lambda R$ (by Observation 1) lead to

$$\begin{aligned}&\lambda F^2-\lambda\beta x_i^* F-\beta(F-x_i^*)\\ &=\lambda F^2-\beta(\lambda F-1)x_i^*-\beta F \ge \lambda F^2-\beta F\\ &=F(\lambda F-\beta)>0.\end{aligned}$$

These together with $\alpha e^{-\lambda(w_i-x_i^*)}>0$ imply that when $F \ge R$, the right-hand side of (11) is positive, which gives a contradiction.

Now suppose $\lambda F \le \beta$. Then $\lambda S(x_{-i}^*) \le \beta$, so (iii) of Lemma 2 applies without any additional condition on w_i, so all players must be active. If $\lambda F > \beta$, then (iii) of Lemma 2 applies with the additional condition on w_i.

Theorem 1 sets the equations to compute the unique Nash equilibrium under linex utility function, and gives a characterization of the set of active players. Statements (i) and (ii) suggest that either all players are active or the active ones are the ones with large wealth levels. The following is an immediate corollary of Theorem 1.

Corollary 1. In the unique Nash equilibrium x^* of the stochastic contest with parameters n, R, λ, α, and $w_1 \ge \cdots \ge w_n$, the set of active players is of the form $\{1, \ldots, m\}$ for some $2 \le m \le n$.

Another implication of Theorem 1 concerns the ratio of the total investment $S(x^*)$ to the contest prize R, which is referred to as the *dissipation ratio* in rent-seeking literature. Theorem 1 suggests that under-dissipation occurs in stochastic contests with linex utility functions. Proposition 2 provides a smaller upper bound and a lower bound on the total equilibrium investment.

Proposition 2. In the unique Nash equilibrium x^* of the stochastic contest with parameters n, R, λ, α, and $w_1 \ge \cdots \ge w_n$, the total investment $S(x^*)$ satisfies

$$\frac{e^{\lambda R}-1}{\lambda(e^{\lambda R}+1)} \le \frac{\beta(m-1)}{\lambda(m-\beta)} < S(x^*) < \frac{(m-1)R}{m} \le \frac{(n-1)R}{n} < R,$$

where m is the number of active players.

Proof. Dividing (11) by $F = S(x^*) > 0$, letting $z_i = x_i^*/F$ and $g(t) = \lambda F - \lambda\beta F t - \beta(1-t)$ gives

$$(1-\alpha)\left[R(1-z_i) - F\right] = \alpha e^{-\lambda(w_i - F z_i)} g(z_i). \tag{12}$$

We add these equations over the set of active players $\{1, \ldots, m\}$ to obtain

$$(1-\alpha)\left[R(m-1) - mF\right] = \alpha \sum_{i=1}^{m} e^{-\lambda(w_i - F z_i)} g(z_i). \tag{13}$$

If $F \ge (m-1)R/m$, the left-hand side of (13) is nonpositive. By (7), $g(z_i) > 0$ if $\lambda F \ge 1$. Alternatively if $\lambda F < 1$, then $g(x)$ is increasing in x. For any player $j \in \operatorname{argmax}_{1\le i\le m} z_i$, it must be true that $z_j \ge 1/m$, since otherwise $\sum_{i=1}^{m} z_i < 1$, Then $F\ge(m-1)R/m$ implies $R(1-z_j) - F \le (m-1)R/m - F \le 0$, so the left-hand side of (12) for player j must be nonpositive, requiring $g(z_j) \le 0\$.As\$g(x)$ is increasing in x, $g(z_i) \le g(z_j) \le 0 \ forall \ 1 \le i \le m$. However,

$$\begin{aligned}\sum_{i=1}^{m} g(z_i) &= m\lambda F - \lambda\beta F - \beta(m-1) = (m-\beta)\lambda F - \beta(m-1)\\ &> \left(\frac{m-1}{m}\right)\left[(m-\beta)\lambda R - \beta m\right] = \left(\frac{m-1}{m}\right)\left[(m-1)\lambda R + e^{-\lambda R}\lambda R - m + m e^{-\lambda R}\right]\\ &= \left(\frac{m-1}{m}\right) h_m(\lambda R) > 0,\end{aligned}$$

where $h_m(x)$ is defined in Observation 3 and positivity follows from (8). Consequently, $\sum_{i=1}^{m} g(z_i) > 0$, which contradicts $g(z_i) \le 0$ for all $1 \le i \le m$. Hence $F \ge (m-1)R/m$ is not possible, and since $m \le n\$and\$(m-1)/m$ is increasing in m, $(m-1)R/m \le (n-1)R/n < R$.

As shown above, if $\lambda F \ge 1$, each $g(z_i) > 0$. On the other hand, if $\lambda F < 1$ and $g(z_k) \le 0$ for some $1 \le k \le m$, then $R(1-z_k) \le F$, implying $R(1-z_j) \le F$ for $j \in \operatorname{argmax}_{1 \le i \le m} z_i$ and so $g(z_j) \le 0$. Since $\lambda F < 1$ implies that $g(t)$ is increasing in t, $g(z_i) \le 0$ for all $1 \le i \le m$. The latter cannot be true since the left-hand side of (13) is $(1-\alpha)[R(m-1) - mF] > 0$ as shown in the first part of this proof. Hence $g(z_i) > 0$ for all $1 \le i \le m$ must hold also when $\lambda F < 1$. Consequently,

$$\sum_{i=1}^{m} g(z_i) = (m-\beta)\lambda F - \beta(m-1) > 0,$$

implying $\lambda F > \beta(m-1)/(m-\beta)$. We obtain the leftmost bound by observing that $m \ge 2$ and $(m-1)/(m-\beta)$ is increasing in m, replacing $\beta = 1 - e^{-\lambda R}$, and multiplying both the numerator and the denominator by $e^{\lambda R}$.

In order to see the implications of these bounds on the dissipation ratio, we divide all by R and get

$$\frac{e^{\lambda R} - 1}{\lambda R(e^{\lambda R} + 1)} < \frac{S(x^*)}{R} < \frac{(n-1)}{n} < 1.$$

The lower bound converges to zero as λR increases, since the function $(e^x - 1)/[x(e^x + 1)]$ has derivative

$$\frac{2xe^x - e^{2x} + 1}{x^2(e^x+1)^2} = \frac{-(e^x - x)^2 + x^2 + 1}{x^2(e^x+1)^2} \le \frac{-(1 + x^2/2)^2 + x^2 + 1}{x^2(e^x+1)^2} = \frac{-x^4}{4x^2(e^x+1)^2} \le 0.$$

Proposition 3. In the unique Nash equilibrium x^* of the stochastic contest with parameters n, R, λ, α, and $w_1 \ge w_2 \ge \cdots \ge w_n$, if $w_i > w_j$, then $x_i^* > x_j^*$.

Proof. The proof of Proposition 2 showed that $g(z_i) > 0$, so $R(1-z_i) > F = S(x^*)$ for each $i = 1, \ldots, m$. Then (12) can be reorganized as follows

$$e^{\lambda w_i} = \left(\frac{\alpha}{1-\alpha}\right) e^{\lambda F z_i} \left[\frac{g(z_i)}{R(1-z_i) - F}\right].$$

The left-hand side of this equality is increasing in w_i. The derivative of the ratio on the right-hand side with respect to z_i is

$$\frac{F(\lambda\beta F - \beta - \lambda\beta R + \lambda R)}{[R(1-z_i) - F]^2} > \frac{F(\beta^2/(2-\beta) - \beta - \lambda\beta R + \lambda R)}{[R(1-z_i) - F]^2}$$
$$= \frac{(1-\beta)F(\lambda R(2-\beta) - 2\beta)}{(2-\beta)[R(1-z_i) - F]^2} > 0,$$

where the first inequality uses the lower bound in Proposition 2, and positivity follows from (8) with $m = 2$. Hence the right-hand side of the reorganized equilibrium condition above is increasing in z_i, and therefore, for any F, larger z_i should be allocated to the player with larger wealth w_i.

Proposition 3 states that wealthier players invest more in the contest. This in turn implies that the probability $p_i(x^*)$ of winning the contest for a wealthier player is higher, so we come across yet another instance of the rich being more likely to get richer.

The next two sections explore further properties of the Nash equilibrium in the special cases involving arbitrarily many identical players and two players with different wealth levels respectively.

4. Contests with Identical Players

This section restricts attention to contests among players with equal initial wealth $w_i = w > 0$. Corollary 2 applies Proposition 2 to this special case, which is then used to refine Theorem 1.

Corollary 2. In the unique Nash equilibrium x^* of the stochastic contest with parameters n, R, λ, α and $w_1 = \ldots = w_n = w$, all players are active and the total investment $S(x^*)$ satisfies

$$\frac{(n-1)\beta}{\lambda(n-\beta)} < S(x^*) < \frac{(n-1)R}{n} < R.$$

Proof. By Theorem 1, if $\lambda F = \lambda S(x^*) \le \beta$, all players must be active; otherwise, players with wealth violating the additional condition in (ii) are active, but when $w_i = w \ for all\ i$, the condition holds either for all or none of the players. As stated at the beginning of the same theorem, there are always at least two active players, so with identical players, all must be active, or equivalently $m = n$. Plugging this in the inequalities in Proposition 2 completes the proof.

Theorem 2. In the unique Nash equilibrium of the stochastic contest with parameters n, R, λ, α and $w_1 = \ldots = w_n = w$, all players invest equally and the total investment F is the unique solution of the equation

$$(1-\alpha)\left[R(n-1) - nF\right] = \alpha e^{-\lambda(w-F/n)}\left[\lambda(n-\beta)F - \beta(n-1)\right]. \tag{14}$$

Proof. As shown in the proof of Proposition 2 $g(z_i) > 0$, so $R(1-z_i) > F = S(x^*)$ for each $i = 1, \ldots, m = n$, and (12) is equivalent to

$$e^{\lambda w} = \left(\frac{\alpha}{1-\alpha}\right) e^{\lambda F z_i}\left[\frac{g(z_i)}{R(1-z_i) - F}\right].$$

The proof of Proposition 3 showed that the right-hand side of this equality is increasing in z_i, so for any given $F = S(x^*)$, z_i is unique, assuring $z_i = z$ for all $i = 1, \ldots, n$ and since $z_i = x_i^* / S(x^*)\$,\$z_i = 1/n$. Letting $x_i = F/n$ and canceling F from both sides reduces (11) to (14), which has a unique solution, since its left-hand side is decreasing in F, and the right-hand side is increasing in F. A solution satisfying the bounds in Corollary 2 exists because at $F = 0$,

$$(1-\alpha)\left[R(n-1) - nF\right] = (1-\alpha)R(n-1) > 0,$$

$$\alpha e^{-\lambda(w-F/n)}\left[\lambda(n-\beta)F - \beta(n-1)\right] = -\alpha e^{-\lambda w}\beta(n-1) < 0,$$

and at $F = (n-1)R/n$

$$(1-\alpha)\left[R(n-1)-nF\right]=0,$$

$$\alpha e^{-\lambda(w-F/n)}\left[\lambda(n-\beta)F-\beta(n-1)\right]=-\alpha e^{-\lambda(w-(n-1)R/n^2)}\left(\frac{n-1}{n}\right)h_n(\lambda R)>0,$$

where $h_n(x)$ is defined and shown to be positive for $x>0$ in Observation 3.

Theorem 2 reduces the problem of finding the Nash equilibrium to solving the nonlinear equation (14) in the special case with identical players.

Proposition 4. In the stochastic contest among players endowed with equal wealth, the equilibrium investment of players increases as the initial wealth w increases and decreases as the utility parameter α increases.

Proof. The condition (14) can be reorganized as follows:

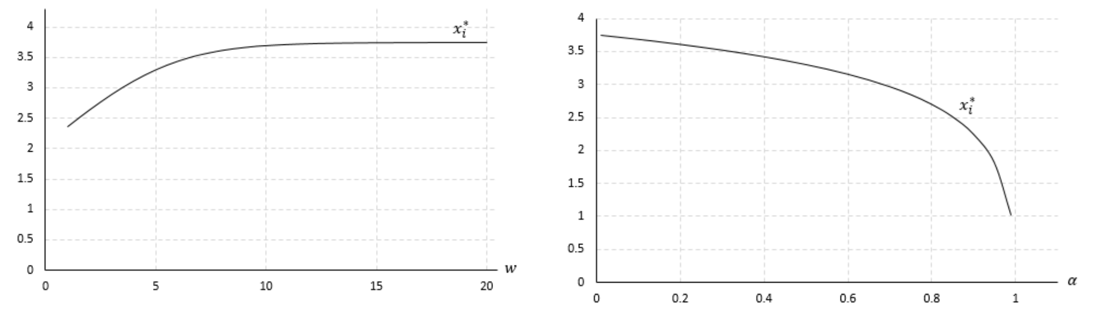
$$\left(\frac{1}{\alpha}-1\right)e^{\lambda w}=e^{\lambda F/n}\left[\frac{\lambda(n-\beta)F-\beta(n-1)}{R(n-1)-nF}\right].$$

The left-hand side of this equality is increasing in w and decreasing in α. In the right-hand side, $e^{\lambda F/n}$ and $\lambda(n-\beta)F-\beta(n-1)$ are increasing in F, $R(n-1)-nF$ is decreasing in F, and each of these expressions is positive as suggested by Corollary 2; therefore, the right-hand side is increasing in F. Consequently, F and so F/n must be increasing in w and decreasing in α.

As the wealth w increases or the utility parameter α decreases, players become less risk averse, since the coefficient of absolute risk aversion decreases. Proposition 4 states that as a consequence of any of these changes, players invest more in the contest as they become more tolerant to taking risks. We proved this result directly using the equilibrium conditions for the linex utility functions; it can be recovered also by using Proposition 2 in [12].

The following example illustrates the results of this section.

Example 1. Consider a contest with four players, prize $R=20$, and utility parameter $\lambda=0.5$. Figure 1. shows the effects of varying the wealth w and the utility parameter α on the equilibrium investment of players. Consistently with Proposition 4, we observe that equilibrium increases as wealth increases, and decreases as the weight α on the nonlinear (exponential) component of the utility function increases. This example also illustrates that players can choose to invest more than their initial wealth in the Nash equilibrium, e.g., $x_i^*=2.365$ for $\alpha=0.5$ and $w=1$.



a) Effect of increasing w (b) Effect of increasing α

Figure 1. Effects of increasing w and α on equilibrium investment x_i^* in contests with identical player

5. Two-Player Contests

This section is devoted primarily to a numerical experiment involving two players. As the previous section solved the problem for arbitrarily many identical players, here we focus on the case where two players initially have different amounts of wealth, $w_1 = w + \delta, w_2 = w$, and $\delta > 0$. The following corollary applies Theorem 1 to this case.

Corollary 3. In the unique Nash equilibrium of the two-player stochastic contest with parameters R, λ, α, $w_1 = w + \delta$, $w_2 = w$, and $\delta > 0$, both players invest a positive amount and their equilibrium investments x_1^* and x_2^* uniquely solve the following system of nonlinear equations

$$(1-\alpha)\left[Rx_2 - (x_1 + x_2)^2 \right] = \alpha e^{-\lambda(w+\delta-x_1)} \left[\lambda(x_1 + x_2)^2 - \lambda\beta x_1(x_1 + x_2) - \beta x_2 \right], \quad (15)$$

$$(1-\alpha)\left[Rx_1 - (x_1 + x_2)^2 \right] = \alpha e^{-\lambda(w-x_2)} \left[\lambda(x_1 + x_2)^2 - \lambda\beta x_2(x_1 + x_2) - \beta x_1 \right]. \quad (16)$$

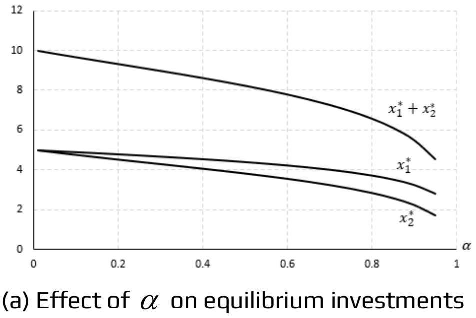
Letting $F = x_1 + x_2$ and $z = x_1 / F$ (so that $1 - z = x_2 / F$) in (15)-(16) reduces the problem of computing the Nash equilibrium of a two-player contest to finding $F \geq 0$ and $z \in [0,1]$ that satisfy the following two nonlinear equations

$$(1-\alpha)\left[R(1-z) - F \right] = \alpha e^{-\lambda(w+\delta-Fz)} \left[\lambda F - \lambda\beta Fz - \beta(1-z) \right], \quad (17)$$

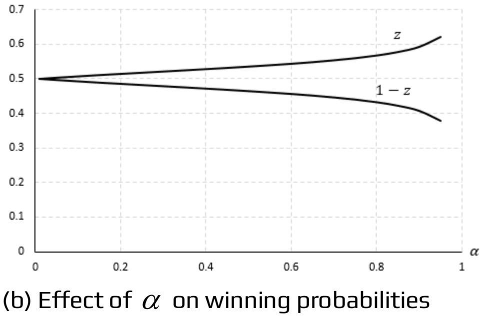
$$(1-\alpha)\left[Rz - F \right] = \alpha e^{-\lambda(w-F(1-z))} \left[\lambda F - \lambda\beta F(1-z) - \beta z \right]. \quad (18)$$

We illustrate the sensitivity of the Nash equilibrium to changes in parameters α, w, and δ through a numerical experiment. We recall that an increase in α corresponds to an increase in the risk-sensitivity of the two players (but at different degrees since they have different wealth), an increase in w represents an equal increase in their wealth, and an increase in δ means an increase only in the wealth of player 1. The latter two have also implications on the degree at which players are risk averse; specifically, an increase in w makes both players less risk averse whereas an increase in δ makes only player 1 less risk averse. Independently of whether the effect is on only one or both of the players' risk sensitivity, both players' investments are affected by these changes because of the game theoretical nature of the problem.

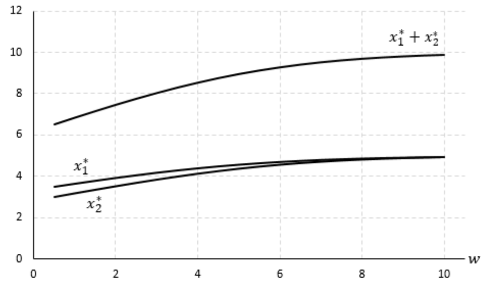
Example 2. Consider the contest between two players with prize $R = 20$, and utility parameter $\lambda = 0.5$. Figure 2 displays the changes in the equilibrium investments and winning probabilities of players as α, w, and δ increases.



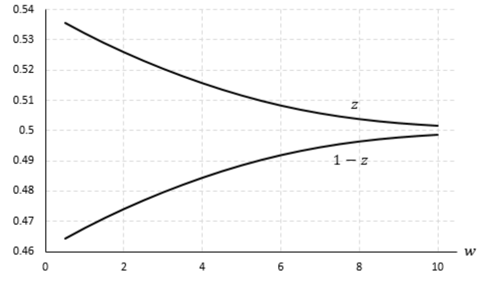
(a) Effect of α on equilibrium investments



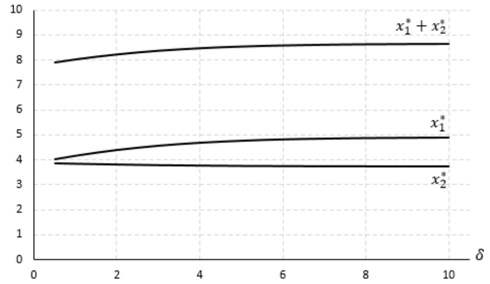
(b) Effect of α on winning probabilities



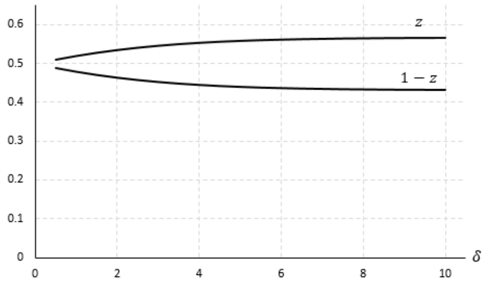
(c) Effect of w on equilibrium investments



(d) Effect of w on winning probabilities



(e) Effect of δ on equilibrium investments



(f) Effect of δ on winning probabilities

Figure 2. Effects of increasing α, w, and δ on equilibrium investments and winning probabilities in two-player contests

An increase in α, which is the weight of the nonlinear component of the utility function, increases the coefficient of absolute risk aversion of both players; however, since their initial wealth is different, two players experience different levels of increase in their risk aversion. For α close to zero, the coefficient of absolute risk aversion is very small, so players are close to risk neutral and their choices are minimally affected by the discrepancy between their wealth. Figure 2 (a) shows that as α increases, their investments decrease while the gap between their investments increases. On the other hand, Figure 2 (b) shows that the decreases in x_1^* and x_2^* are such that the winning probability $z = x_1^* / (x_1^* + x_2^*)$ of player 1 increases whereas the winning probability $1-z$ of player 2 decreases.

Given α and δ constant, an increase in w represents an equal increase in the wealth of two players, which, in turn, affects the degree of risk aversion exhibited by each player. We observe in Figure 2 (c) that such a change occurs, both players invest more in the Nash equilibrium, which can be explained by decreasing absolute risk aversion. As w gets larger, the difference between their investments appears to be vanishing, since the effect of the constant gap between their wealth on their preferences decreases. Figure 2 (d) suggests that an equal increase in wealth affects the winning probability of player 1 adversely and that of player 2 favorably, even though the latter always remains below the former.

Finally, for fixed α and w, the effects of increasing the gap δ between the wealth of two players affects the equilibrium investments in opposite directions. Specifically, as δ increases, the equilibrium investment of player 1 increases whereas the investment of player 2 decreases. The increase in x_1^* overweighs the decrease in x_2^*, so the total investment $x_1^* + x_2^*$ increases. These changes are displayed in Figure 2 (e). The changes in winning probabilities follow straightforwardly from the changes in equilibrium investments. Since x_1^* increases and x_2^* decreases, the ratio $z = x_1^* / (x_1^* + x_2^*)$ increases and $1-z$ decreases, meaning the winning probability of player 1 increases while the winning probability of player 2 decreases, as shown in

Figure 2 (f). This makes intuitive sense, since the gap between the wealth levels of players increases, player 1 enjoys a more advantageous status, she is wealthier and so by decreasing absolute risk aversion, much more tolerant to risk than her competitor. Consequently, she invests more in the risky contest, further improving her chance of winning.

6. Conclusion

In this paper, we explored the Nash equilibrium in a class of winner-takes-all stochastic contests among players with linex utility functions. We derived the equilibrium conditions and by delving into these conditions, obtained bounds on the total equilibrium investment, and elicited the effects of wealth and that of the weight on the nonlinear (so risk-sensitive) part of the utility function on the equilibrium. In particular, for the general case, we showed that there is a cut characterization of the players that invest a positive amount with respect to wealth, and wealthier players invest more, making them more likely to win the contest. For the case with identical players, we showed that a larger wealth and a smaller weight on the nonlinear utility term implies a larger equilibrium investment, which is also intuitive since both these changes entail higher risk tolerance.

Potential future research directions include employing different types of utility functions or even alternative risk-sensitive criteria such as those based on mean and variance, and conditional value at risk, considering various forms of winning probability or contest mechanisms other than winner-takes-all, and exploring the dynamic versions where players get involved in successive contests and the outcome of these contests affect their future decisions by increasing or decreasing their wealth and consequently their attitude toward risk.

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